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Common fixed points of Ćirić-type contractive mappings in two ordered generalized metric spaces

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Abstract

In this paper, using the setting of two ordered generalized metric spaces, a unique common fixed point of four mappings satisfying a generalized contractive condition is obtained. We also present an example to demonstrate the results presented herein.

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1 Introduction and preliminaries

The study of a unique common fixed point of given mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [1] generalized the concept of a metric in which a real number is assigned to every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa *et al.* [2–5] obtained some fixed point theorems for some mappings satisfying different contractive conditions. The existence of common fixed points in generalized metric spaces was initiated by Abbas and Rhoades [6] (see also [7] and [8]). For further study of common fixed points in generalized metric spaces, we refer to [9–12] and references mentioned therein. Abbas *et al.* [13] showed the existence of coupled common fixed points in two generalized metric spaces (for more results on couple fixed points, see also [14–21]).

The existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [22] and further studied by Nieto and Lopez [23]. Subsequently, several interesting and valuable results have appeared in this direction [24–30].

The aim of this paper is to study common fixed point of four mappings that satisfy the generalized contractive condition in two ordered generalized metric spaces.

In the sequel, \mathbb{R} , \mathbb{R}^+ and \mathbb{N} denote the set of real numbers, the set of nonnegative integers and the set of positive integers respectively. The usual order on \mathbb{R} (respectively, on \mathbb{R}^+) will be indistinctly denoted by \leq or by \geq .

In [1], Mustafa and Sims introduced the following definitions and results:

Definition 1.1 Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}^+$ satisfies the following conditions:

- (a) $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$;
- (b) $0 < G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$;

- (c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
- (d) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of $x, y, z \in X$ (symmetry);
- (e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G -metric on X and (X, G) is called a G -metric space.

Definition 1.2 A sequence $\{x_n\}$ in a G -metric space X is called:

- (1) a G -Cauchy sequence if, for any $\varepsilon > 0$, there exists $n_0 \in N$ (the set of natural numbers) such that, for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;
- (2) G -convergent if, for any $\varepsilon > 0$, there exist $x \in X$ and $n_0 \in N$ such that, for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$;
- (3) A G -metric space X is said to be G -complete if every G -Cauchy sequence in X is G -convergent in X .

It is known that $\{x_n\}$ is G -convergent to a point $x \in X$ if and only if $G(x_m, x_n, x) \rightarrow 0$ as $n, m \rightarrow \infty$.

Proposition 1.3 [1] Let X be a G -metric space. Then the following items are equivalent:

- (1) A sequence $\{x_n\}$ in X is G -convergent to a point $x \in X$;
- (2) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow \infty$;
- (3) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (4) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.4 A G -metric on X is said to be *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5 Every G -metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad (1.1)$$

for all $x, y \in X$.

For a symmetric G -metric, we have

$$d_G(x, y) = 2G(x, y, y) \quad (1.2)$$

for all $x, y \in X$. However, if G is non-symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \quad (1.3)$$

for all $x, y \in X$. It is obvious that

$$G(x, x, y) \leq 2G(x, y, y)$$

for all $x, y \in X$.

Now, we give an example of a non-symmetric G -metric.

Table 1 G-metric

(x, y, z)	$G(x, y, z)$
$(1, 1, 1), (2, 2, 2)$	0
$(1, 1, 2), (1, 2, 1), (2, 1, 1)$	0.5
$(1, 2, 2), (2, 1, 2), (2, 2, 1)$	1.0

Example 1.6 Let $X = \{1, 2\}$ and $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a mapping defined by Table 1.

Note that G satisfies all the axioms of a generalized metric, but $G(x, x, y) \neq G(x, y, y)$ for two distinct points $x, y \in X$.

Definition 1.7 Let f and g be self-mappings on a set X . If $w = fx = gx$ for some $x \in X$, then the point x is called a *coincidence point* of f and g and w is called a *point of coincidence* of f and g .

Definition 1.8 [31] Let f and g be self-mappings on a set X . Then f and g are said to be *weakly compatible* if they commute at every coincidence point.

Definition 1.9 [8] Let X be a G -metric space and f, g be self-mappings on X . Then f and g are said to be *R -weakly commuting* if there exists a positive real number R such that $G(fgx, fgx, gfx) \leq RG(fx, fx, gx)$ for all $x \in X$.

The maps f and g are R -weakly commuting on X if and only if they commute at their coincidence points.

Recall that two mappings f and g on a G -metric space X are said to be *compatible* if, for a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G -convergent to some $t \in X$,

$$\lim_{n \rightarrow \infty} G(fgx_n, fgx_n, gfx_n) = 0.$$

Definition 1.10 Let X be a nonempty set. Then (X, \preceq, G) is called an *ordered generalized metric space* if the following conditions hold:

- (a) G is a generalized metric on X ;
- (b) \preceq is a partial order on X .

Definition 1.11 Let (X, \preceq) be a partially ordered set. Then two points $x, y \in X$ are said to be *comparable* if $x \preceq y$ or $y \preceq x$.

Definition 1.12 [24] Let (X, \preceq) be a partially ordered set. A self-mapping f on X is said to be *dominating* if $x \preceq fx$ for all $x \in X$.

Example 1.13 [24] Let $X = [0, 1]$ be endowed with usual ordering and $f : X \rightarrow X$ be a mapping defined by $fx = \sqrt[n]{x}$ for some $n \in \mathbb{N}$. Since $x \leq x^{\frac{1}{n}} = fx$ for all $x \in X$, f is a dominating mapping.

Definition 1.14 Let (X, \preceq) be a partially ordered set. A self-mapping f on X is said to be *dominated* if $fx \preceq x$ for all $x \in X$.

Example 1.15 Let $X = [0, 1]$ be endowed with usual ordering and $f : X \rightarrow X$ be a mapping defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, f is a dominated mapping.

Definition 1.16 A subset \mathcal{K} of a partially ordered set X is said to be *well-ordered* if every two elements of \mathcal{K} are comparable.

2 Common fixed point theorems

In [32], Kannan proved a fixed point theorem for a single valued self-mapping T on a metric space X satisfying the following property:

$$d(Tx, Ty) \leq h\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$, where $h \in [0, \frac{1}{2})$. If a self-mapping T on a metric space X satisfies the following property:

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + e[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$, where $a, b, c, e \geq 0$ with $a + b + c + 2e < 1$, then T has a unique fixed point provided that X is T -orbitally complete (for related definitions and results, we refer to [33]).

Afterwards, Ćirić [34] obtained a fixed point result for a mapping satisfying the following property:

$$d(Tx, Ty) \leq q \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}$$

for all $x, y \in X$, where $0 \leq q < 1$.

In this section, we show the existence of a unique common fixed point of four mappings satisfying Ćirić-type contractive condition in the framework of two ordered generalized metric spaces.

Now, we start with the following result:

Theorem 2.1 Let (X, \preceq) be a partially ordered set and G_1, G_2 be two G -metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$ for all $x, y, z \in X$ with a complete metric G_1 on X . Suppose that f, g, S and T are self-mappings on X satisfying the following properties:

$$G_1(fx, fx, gy) \leq k \max \{ G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), [G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]/2 \} \quad (2.1)$$

and

$$G_1(fx, gy, gy) \leq k \max \{ G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), [G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)]/2 \} \quad (2.2)$$

for all comparable $x, y \in X$, where $k \in [0, 1)$. Suppose that $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings. If, for any nonincreasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all $n \in \mathbb{N}$, $y_n \rightarrow u$ implies that $u \preceq x_n$ and either

(a) f, S are compatible, f or S is continuous and g, T are weakly compatible

or

(b) g, T are compatible, g or T is continuous and f, S are weakly compatible, then f, g, S and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well-ordered if and only if f, g, S and T have one and only one common fixed point.

Proof Let x_0 be an arbitrary point in X . Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n} = gx_{2n} = Sx_{2n+1}, \quad y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$$

for all $n \geq 0$. By the given assumptions, we have

$$x_{2n+2} \leq Tx_{2n+2} = fx_{2n+1} \leq x_{2n+1},$$

$$x_{2n+1} \leq Sx_{2n+1} = gx_{2n} \leq x_{2n}.$$

Thus, for all $n \geq 0$, we have $x_{n+1} \leq x_n$. Suppose that $G_1(y_{2n}, y_{2n+1}, y_{2n+1}) > 0$ for all $n \geq 0$. If not, then, for some $m \geq 0$, $y_m = y_{m+1}$. Indeed, if $m = 2k$, then $y_{2k} = y_{2k+1}$ and from (2.1), it follows that

$$\begin{aligned} & G_1(y_{2k+1}, y_{2k+1}, y_{2k+2}) \\ &= G_1(fx_{2k+1}, fx_{2k+1}, gx_{2k+2}) \\ &\leq k \max \{ G_2(Sx_{2k+1}, Sx_{2k+1}, Tx_{2k+2}), G_2(fx_{2k+1}, fx_{2k+1}, Sx_{2k+1}), \\ &\quad G_2(gx_{2k+2}, gx_{2k+2}, Tx_{2k+2}), \\ &\quad [G_2(fx_{2k+1}, fx_{2k+1}, Tx_{2k+2}) + G_2(gx_{2k+2}, gx_{2k+2}, Sx_{2k+1})]/2 \} \\ &= k \max \{ G_2(y_{2k}, y_{2k}, y_{2k+1}), G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\quad [G_2(y_{2k+1}, y_{2k+1}, y_{2k+1}) + G_2(y_{2k+2}, y_{2k+2}, y_{2k})]/2 \} \\ &\leq k \max \{ G_2(y_{2k}, y_{2k}, y_{2k+1}), G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\quad [G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G_2(y_{2k+1}, y_{2k+1}, y_{2k})]/2 \} \\ &\leq k \max \{ G_1(y_{2k}, y_{2k}, y_{2k+1}), G_1(y_{2k+1}, y_{2k+1}, y_{2k}), G_1(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\quad [G_1(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G_1(y_{2k+1}, y_{2k+1}, y_{2k})]/2 \} \\ &= kG_1(y_{2k+2}, y_{2k+2}, y_{2k+1}). \end{aligned} \tag{2.3}$$

Again, from (2.2), it follows that

$$\begin{aligned} & G_1(y_{2k+1}, y_{2k+2}, y_{2k+2}) \\ &= G_1(fx_{2k+1}, gx_{2k+2}, gx_{2k+2}) \\ &\leq k \max \{ G_2(Sx_{2k+1}, Tx_{2k+2}, Tx_{2k+2}), G_2(fx_{2k+1}, Sx_{2k+1}, Sx_{2k+1}), \\ &\quad G_2(gx_{2k+2}, Tx_{2k+2}, Tx_{2k+2}), \\ &\quad [G_2(fx_{2k+1}, Tx_{2k+2}, Tx_{2k+2}) + G_2(gx_{2k+2}, Sx_{2k+1}, Sx_{2k+1})]/2 \} \end{aligned}$$

$$\begin{aligned}
 &= k \max \{ G_2(y_{2k}, y_{2k+1}, y_{2k+1}), G_2(y_{2k+1}, y_{2k}, y_{2k}), G_2(y_{2k+2}, y_{2k+1}, y_{2k+1}), \\
 &\quad [G_2(y_{2k+1}, y_{2k+1}, y_{2k+1}) + G_2(y_{2k+2}, y_{2k}, y_{2k})]/2 \} \\
 &\leq k \max \{ G_2(y_{2k}, y_{2k+1}, y_{2k+1}), G_2(y_{2k+1}, y_{2k}, y_{2k}), G_2(y_{2k+2}, y_{2k+1}, y_{2k+1}), \\
 &\quad [G_2(y_{2k+2}, y_{2k+1}, y_{2k+1}) + G_2(y_{2k+1}, y_{2k}, y_{2k})]/2 \} \\
 &\leq k \max \{ G_1(y_{2k}, y_{2k}, y_{2k+1}), G_2(y_{2k+1}, y_{2k}, y_{2k}), G_1(y_{2k+2}, y_{2k+1}, y_{2k+1}), \\
 &\quad [G_1(y_{2k+2}, y_{2k+1}, y_{2k+1}) + G_1(y_{2k+1}, y_{2k}, y_{2k})]/2 \} \\
 &= k G_1(y_{2k+2}, y_{2k+1}, y_{2k+1}). \tag{2.4}
 \end{aligned}$$

Thus (2.3) and (2.4) imply that

$$G_1(y_{2k+2}, y_{2k+1}, y_{2k+1}) \leq k^2 G_1(y_{2k+2}, y_{2k+1}, y_{2k+1})$$

and so $y_{2k+1} = y_{2k+2}$ since $k^2 < 1$.

Similarly, if $m = 2k + 1$, then one can easily obtain $y_{2k+2} = y_{2k+3}$. Thus $\{y_n\}$ becomes a constant sequence and y_{2n} serves as the common fixed point of f , g , S and T .

Suppose that $G_1(y_{2n}, y_{2n+1}, y_{2n+1}) > 0$ for all $n \geq 0$.

If $n \in \mathbb{N}$ is even, then $n = 2k$ for some $k \in \mathbb{N}$; then it follows from (2.1) that

$$\begin{aligned}
 &G_1(y_{n+1}, y_{n+1}, y_n) \\
 &= G_1(y_{2k+1}, y_{2k+1}, y_{2k}) \\
 &= G_1(fx_{2k+1}, fx_{2k+1}, gx_{2k}) \\
 &\leq k \max \{ G_2(Sx_{2k+1}, Sx_{2k+1}, Tx_{2k}), G_2(fx_{2k+1}, fx_{2k+1}, Sx_{2k+1}), \\
 &\quad G_2(gx_{2k}, gx_{2k}, Tx_{2k}), [G_2(fx_{2k+1}, fx_{2k+1}, Tx_{2k}) + G_2(gx_{2k}, gx_{2k}, Sx_{2k+1})]/2 \} \\
 &= k \max \{ G_2(y_{2k}, y_{2k}, y_{2k-1}), G_2(y_{2k+1}, y_{2k+1}, y_{2k}), \\
 &\quad G_2(y_{2k}, y_{2k}, y_{2k-1}), [G_2(y_{2k+1}, y_{2k+1}, y_{2k-1}) + G_2(y_{2k}, y_{2k}, y_{2k})]/2 \} \\
 &\leq k \max \{ G_2(y_{2k}, y_{2k}, y_{2k-1}), G_2(y_{2k+1}, y_{2k+1}, y_{2k}), \\
 &\quad [G_2(y_{2k+1}, y_{2k+1}, y_{2k}) + G_2(y_{2k}, y_{2k}, y_{2k-1})]/2 \} \\
 &\leq k \max \{ G_1(y_n, y_n, y_{n-1}), G_1(y_{n+1}, y_{n+1}, y_n) \},
 \end{aligned}$$

which implies that

$$G_1(y_{n+1}, y_{n+1}, y_n) \leq k G_1(y_n, y_n, y_{n-1}).$$

If $n \in \mathbb{N}$ is odd, then $n = 2k + 1$ for some $k \in \mathbb{N}$. Again, it follows from (2.1) that

$$\begin{aligned}
 &G_1(y_{n+1}, y_{n+1}, y_n) \\
 &= G_1(y_{2k+2}, y_{2k+2}, y_{2k+1}) \\
 &= G_1(fx_{2k+2}, fx_{2k+2}, gx_{2k+1}) \\
 &\leq k \max \{ G_2(Sx_{2k+2}, Sx_{2k+2}, Tx_{2k+1}), G_2(fx_{2k+2}, fx_{2k+2}, Sx_{2k+2}), \\
 &\quad G_2(gx_{2k+1}, gx_{2k+1}, Tx_{2k+1}), [G_2(fx_{2k+2}, fx_{2k+2}, Tx_{2k+1}) + G_2(gx_{2k+1}, gx_{2k+1}, Sx_{2k+2})]/2 \}
 \end{aligned}$$

$$\begin{aligned}
 & G_2(gx_{2k+1}, gx_{2k+1}, Tx_{2k+1}), \\
 & [G_2(fx_{2k+2}, fx_{2k+2}, Tx_{2k+1}) + G_2(gx_{2k+1}, gx_{2k+1}, Sx_{2k+2})]/2\} \\
 = & k \max\{G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\
 & G_2(y_{2k+1}, y_{2k+1}, y_{2k}), [G_2(y_{2k+2}, y_{2k+2}, y_{2k}) + G_2(y_{2k+1}, y_{2k+1}, y_{2k+1})]/2\} \\
 \leq & k \max\{G_2(y_{2k+1}, y_{2k+1}, y_{2k}), G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\
 & [G_2(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G_2(y_{2k+1}, y_{2k+1}, y_{2k})]/2\} \\
 \leq & k \max\{G_1(y_{2k+1}, y_{2k+1}, y_{2k}), G_1(y_{2k+2}, y_{2k+2}, y_{2k+1})\} \\
 = & k \max\{G_1(y_n, y_n, y_{n-1}), G_1(y_{n+1}, y_{n+1}, y_n)\},
 \end{aligned}$$

that is,

$$G_1(y_{n+1}, y_{n+1}, y_n) \leq k G_1(y_n, y_n, y_{n-1})$$

for all $n \in \mathbb{N}$. Continuing the above process, we have

$$G_1(y_{n+1}, y_{n+1}, y_n) \leq k^n G_1(y_1, y_1, y_0)$$

for all $n \in \mathbb{N}$. Thus, for all $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
 & G_1(y_m, y_m, y_n) \\
 & \leq G_1(y_n, y_{n+1}, y_{n+1}) + G_1(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G_1(y_{m-1}, y_m, y_m) \\
 & \leq k^n G_1(y_0, y_1, y_1) + k^{n+1} G_1(y_0, y_1, y_1) + \cdots + k^{m-1} G_1(y_0, y_1, y_1) \\
 & = k^n G_1(y_0, y_1, y_1) \sum_{i=0}^{m-n-1} k^i \\
 & \leq \frac{k^n}{1-k} G_1(y_0, y_1, y_1)
 \end{aligned}$$

and so $G_1(y_n, y_m, y_m) \rightarrow 0$ as $m, n \rightarrow \infty$. Hence $\{y_n\}$ is a G -Cauchy sequence in X . Since X is G_1 -complete, there exists a point $z \in X$ such that $\lim_{n \rightarrow \infty} y_n = z$. Consequently, we have

$$\lim_{n \rightarrow \infty} y_{2n+1} = \lim_{n \rightarrow \infty} fx_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n+2} = z$$

and

$$\lim_{n \rightarrow \infty} y_{2n} = \lim_{n \rightarrow \infty} gx_{2n} = \lim_{n \rightarrow \infty} Sx_{2n+1} = z.$$

If S is continuous and $\{f, S\}$ is compatible, then

$$\lim_{n \rightarrow \infty} S^2 x_{2n+1} = Sz,$$

$$\lim_{n \rightarrow \infty} fSx_{2n+1} = \lim_{n \rightarrow \infty} Sfx_{2n+1} = Sz.$$

Since $Sx_{2n+1} = gx_{2n} \leq x_{2n}$, (2.1) gives

$$\begin{aligned} & G_1(fSx_{2n+1}, fSx_{2n+1}, gx_{2n}) \\ & \leq k \max \{ G_2(SSx_{2n+1}, SSx_{2n+1}, Tx_{2n}), G_2(fSx_{2n+1}, fSx_{2n+1}, SSx_{2n+1}), \\ & \quad G_2(gx_{2n}, gx_{2n}, Tx_{2n}), \\ & \quad [G_2(fSx_{2n+2}, fSx_{2n+2}, Tx_{2n}) + G_2(gx_{2n}, gx_{2n}, SSx_{2n+1})]/2 \}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} G_1(Sz, Sz, z) & \leq k \max \{ G_2(Sz, Sz, z), G_2(Sz, Sz, Sz), G_2(z, z, z), \\ & \quad [G_2(Sz, Sz, z) + G_2(z, z, Sz)]/2 \} \\ & \leq k \max \{ G_1(Sz, Sz, z), [G_1(Sz, Sz, z) + G_1(z, z, Sz)]/2 \} \\ & = \frac{k}{2} [G_1(Sz, Sz, z) + G_1(z, z, Sz)], \end{aligned}$$

which further implies that

$$G_1(Sz, Sz, z) \leq hG_1(z, z, Sz), \quad (2.5)$$

where $h = \frac{k}{2-k}$. Obviously, $0 \leq h < 1$.

Similarly, we obtain

$$G_1(Sz, z, z) \leq hG_1(z, Sz, Sz). \quad (2.6)$$

From (2.5) and (2.6), we have

$$G_1(Sz, Sz, z) \leq h^2 G_1(z, Sz, Sz)$$

and so $Sz = z$ since $0 \leq h^2 < 1$. Since $gx_{2n} \leq x_{2n}$ and $gx_{2n} \rightarrow z$ as $n \rightarrow \infty$ implies $z \leq x_{2n}$, it follows from (2.1) that

$$\begin{aligned} & G_1(fz, fz, gx_{2n}) \\ & \leq k \max \{ G_2(Sz, Sz, Tx_{2n}), G_2(fz, fz, Sz), G_2(gx_{2n}, gx_{2n}, Tx_{2n}), \\ & \quad [G_2(fz, fz, Tx_{2n}) + G_2(gx_{2n}, gx_{2n}, Sz)]/2 \} \\ & = k \max \{ G_2(z, z, Tx_{2n}), G_2(fz, fz, z), G_2(gx_{2n}, gx_{2n}, Tx_{2n}), \\ & \quad [G_2(fz, fz, Tx_{2n}) + G_2(gx_{2n}, gx_{2n}, z)]/2 \}, \end{aligned}$$

which, taking the limit as $n \rightarrow \infty$, gives

$$\begin{aligned} G_1(fz, fz, z) & \leq k \max \{ G_2(z, z, z), G_2(fz, fz, z), G_2(z, z, z), \\ & \quad [G_2(fz, fz, z) + G_2(z, z, z)]/2 \} \\ & \leq kG_1(fz, fz, z). \end{aligned} \quad (2.7)$$

Similarly, we obtain

$$G_1(fz, z, z) \leq kG_1(z, fz, fz). \quad (2.8)$$

Therefore, by using the above two inequalities, we have $fz = z$.

Since $f(X) \subseteq T(X)$, there exists a point $v \in X$ such that $fz = Tv$. Since $v \preceq Tv = fz \preceq z$, it follows from (2.1) that

$$\begin{aligned} G_1(fz, fz, gv) &\leq k \max \{ G_2(Sz, Sz, Tv), G_2(fz, fz, Sz), G_2(gv, gv, Tv), \\ &\quad [G_2(fz, fz, Tv) + G_2(gv, gv, Sz)]/2 \} \\ &= k \max \{ G_2(fz, fz, fz), G_2(fz, fz, fz), G_2(gv, gv, fz), \\ &\quad [G_2(fz, fz, fz) + G_2(gv, gv, fz)]/2 \} \\ &\leq kG_1(fz, gv, gv). \end{aligned} \quad (2.9)$$

Similarly, we get

$$G_1(fz, gv, gv) \leq kG_1(fz, fz, gv). \quad (2.10)$$

Thus (2.9) and (2.10) imply $fz = gv$. Since g and T are weakly compatible, we have $gz = gfv = gTv = Tgv = Tfz = Tz$, and so z is the coincidence point of g and T .

Now, from (2.1), we have

$$\begin{aligned} G_1(z, z, gz) &= G_1(fz, fz, gz) \\ &\leq k \max \{ G_2(Sz, Sz, Tz), G_2(fz, fz, Sz), G_2(gz, gz, Tz), \\ &\quad [G_2(fz, fz, Tz) + G_2(gz, gz, Sz)]/2 \} \\ &= k \max \{ G_2(z, z, gz), G_2(z, z, z), G_2(gz, gz, gz), \\ &\quad [G_2(z, z, gz) + G_2(gz, gz, z)]/2 \} \\ &= k \max \{ G_2(z, z, gz), [G_2(z, z, gz) + G_2(gz, gz, z)]/2 \} \\ &\leq \frac{k}{2} [G_1(z, z, gz) + G_1(gz, gz, z)], \end{aligned}$$

that is,

$$G_1(z, z, gz) \leq hG_1(gz, gz, z), \quad (2.11)$$

where $h = \frac{k}{2-k}$. Obviously, $0 \leq h < 1$. Using (2.2), we have

$$G_1(z, gz, gz) \leq hG_1(z, z, gz). \quad (2.12)$$

Combining the above two inequalities, we get

$$G_1(z, z, gz) \leq h^2 G_1(z, z, gz)$$

and so $z = gz$. Therefore, $fz = gz = Sz = Tz = z$. The proof is similar when f is continuous. Similarly, if (b) holds, then the result follows.

Now, suppose that the set of common fixed points of f, g, S and T is well ordered. We show that a common fixed point of f, g, S and T is unique. Let u be another common fixed point of f, g, S and T . Then, from (2.1), we have

$$\begin{aligned} G_1(z, z, u) &= G_1(fz, fz, gu) \\ &\leq k \max \{ G_2(Sz, Sz, Tu), G_2(fz, fz, Sz), G_2(gu, gu, Tu), \\ &\quad [G_2(fz, fz, Tu) + G_2(gu, gu, Sz)]/2 \} \\ &= k \max \{ G_2(z, z, u), G_2(z, z, z), G_2(u, u, u), \\ &\quad [G_2(z, z, u) + G_2(u, u, z)]/2 \} \\ &= \frac{k}{2} [G_2(z, z, u) + G_2(u, u, z)] \\ &\leq \frac{1}{2} G_1(z, z, u) + \frac{k}{2} G_1(u, u, z), \end{aligned}$$

that is,

$$G_1(z, z, u) \leq k G_1(z, u, u).$$

Similarly, using (2.2), we obtain

$$G_1(z, u, u) \leq k G_1(z, z, u).$$

Combining the above two inequalities, we get

$$G_1(z, z, u) \leq k^2 G_1(z, z, u)$$

and hence $z = u$.

The converse follows immediately. This completes the proof. \square

Example 2.2 Let $X = \{0, 1, 2, 3\}$ be endowed with the usual ordering and G_1, G_2 be two G -metrics on X defined by Table 2. Then G_1 and G_2 are non-symmetric since $G_1(1, 1, 0) \neq$

Table 2 Two G -metrics

(x, y, z)	$G_1(x, y, z)$	$G_2(x, y, z)$
$(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3),$	0	0
$(0, 0, 2), (0, 2, 0), (2, 0, 0), (0, 2, 2), (2, 0, 2), (2, 2, 0),$ $(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 0, 3), (0, 3, 0), (3, 0, 0),$	4	3
$(0, 1, 1), (1, 0, 1), (1, 1, 0), (0, 3, 3), (3, 0, 3), (3, 3, 0),$ $(1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1),$ $(1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 3, 3), (3, 1, 3), (3, 3, 1),$ $(2, 2, 3), (2, 3, 2), (3, 2, 2), (2, 3, 3), (3, 2, 3), (3, 3, 2),$	8	6
$(0, 1, 2), (0, 1, 3), (0, 2, 1), (0, 2, 3), (0, 3, 1), (0, 3, 2),$ $(1, 0, 2), (1, 0, 3), (1, 2, 0), (1, 2, 3), (1, 3, 0), (1, 3, 2),$ $(2, 0, 1), (2, 0, 3), (2, 1, 0), (2, 1, 3), (2, 3, 0), (2, 3, 1),$ $(3, 0, 1), (3, 0, 2), (3, 1, 0), (3, 1, 2), (3, 2, 0), (3, 2, 1),$	8	6

Table 3 Self maps

x	$f(x)$	$g(x)$	$S(x)$	$T(x)$
0	0	0	0	0
1	0	0	2	2
2	0	2	2	3
3	0	0	3	3

Table 4 Dominated and dominating maps

$x \in X$	f is dominated	g is dominated	S is dominating	T is dominating
$x = 0$	$f(0) = 0$	$g(0) = 0$	$0 = S(0)$	$0 = T(0)$
$x = 1$	$f(1) = 0 < 1$	$g(1) = 0 < 1$	$1 < 2 = S(1)$	$1 < 2 = T(1)$
$x = 2$	$f(2) = 0 < 2$	$g(2) = 2$	$2 = S(2)$	$2 < 3 = T(2)$
$x = 3$	$f(3) = 0 < 3$	$g(3) = 0 < 3$	$3 = S(3)$	$3 = T(3)$

$G_1(1, 0, 0)$ and $G_2(1, 1, 0) \neq G_2(1, 0, 0)$ with $G_2(x, y, z) \leq G_1(x, y, z)$ for all $x, y, z \in X$. Let $f, g, S, T : X \rightarrow X$ be the mappings defined by Table 3. Clearly, $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, f, g are dominated mappings and S, T are dominating mappings, see Table 4.

Now, we shall show that for all comparable $x, y \in X$, (2.1) and (2.2) are satisfied with $k = \frac{3}{4} \in [0, 1)$. Note that for all $x, y \in \{0, 1, 3\}$, $G(fx, fx, gy) = G(fx, gy, gy) = 0$ and (2.1), (2.2) are satisfied obviously.

(1) When $x = 0$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 0$, $Ty = 3$ and so

$$\begin{aligned}
 G_1(fx, fx, gy) &= G_1(0, 0, 2) = 4 \\
 &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 2, 3) = \frac{3}{4}G_2(gy, gy, Ty) \\
 &\leq k \max \{G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), \\
 &\quad [G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]/2\}
 \end{aligned}$$

and

$$\begin{aligned}
 G_1(fx, gy, gy) &= G_1(0, 2, 2) = 4 \\
 &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 3, 3) = \frac{3}{4}G_2(gy, Ty, Ty) \\
 &\leq k \max \{G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\
 &\quad [G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)]/2\}.
 \end{aligned}$$

(2) When $x = 1$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 2$, $Ty = 3$ and so

$$\begin{aligned}
 G_1(fx, fx, gy) &= G_1(0, 0, 2) = 4 \\
 &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 2, 3) = \frac{3}{4}G_2(gy, gy, Ty) \\
 &\leq k \max \{G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), \\
 &\quad [G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]/2\}
 \end{aligned}$$

and

$$\begin{aligned} G_1(fx, gy, gy) &= G_1(0, 2, 2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 3, 3) = \frac{3}{4}G_2(gy, Ty, Ty) \\ &\leq k \max \{ G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\ &\quad [G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)]/2 \}. \end{aligned}$$

(3) When $x = 2$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 2$, $Ty = 3$ and so

$$\begin{aligned} G_1(fx, fx, gy) &= G_1(0, 0, 2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 2, 3) = \frac{3}{4}G_2(Sx, Sx, Ty) \\ &\leq k \max \{ G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), \\ &\quad [G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]/2 \} \end{aligned}$$

and

$$\begin{aligned} G_1(fx, gy, gy) &= G_1(0, 2, 2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 3, 3) = \frac{3}{4}G_2(Sx, Ty, Ty) \\ &\leq k \max \{ G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\ &\quad [G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)]/2 \}. \end{aligned}$$

(4) Finally, when $x = 3$ and $y = 2$, then $fx = 0$, $gy = 2$, $Sx = 3$, $Ty = 3$ and so

$$\begin{aligned} G_1(fx, fx, gy) &= G_1(0, 0, 2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 2, 3) = \frac{3}{4}G_2(gy, gy, Ty) \\ &\leq k \max \{ G_2(Sx, Sx, Ty), G_2(fx, fx, Sx), G_2(gy, gy, Ty), \\ &\quad [G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]/2 \} \end{aligned}$$

and

$$\begin{aligned} G_1(fx, gy, gy) &= G_1(0, 2, 2) = 4 \\ &< \frac{3}{4}(6) = \frac{3}{4}G_2(2, 3, 3) = \frac{3}{4}G_2(gy, Ty, Ty) \\ &\leq k \max \{ G_2(Sx, Ty, Ty), G_2(fx, Sx, Sx), G_2(gy, Ty, Ty), \\ &\quad [G_2(fx, Ty, Ty) + G_2(gy, Sx, Sx)]/2 \}. \end{aligned}$$

Thus, for all cases, the contractions (2.1) and (2.2) are satisfied. Hence all of the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point of f , g , S and g .

If we consider the same set equipped with two metrics given by $d_1(x, y) = |x - y|$ and $d_2(x, y) = \frac{1}{2}|x - y|$ for all $x, y \in X$, then for $x = 1$ and $y = 2$, we have

$$\begin{aligned} d_1(fx, gy) &= d_1(0, 2) = 2 \not\leq 2k \\ &\leq k \max \{d_2(2, 3), d_2(0, 2), d_2(2, 3), [d_2(0, 3) + d_2(2, 2)]/2\} \\ &= k \max \{d_2(Sx, Ty), d_2(fx, Sx), d_2(gy, Ty), [d_2(fx, Ty) + d_2(gy, Sx)]/2\} \end{aligned}$$

for any $k \in [0, 1)$. So corresponding results in ordinary metric spaces cannot be applied in this case.

Theorem 2.1 can be viewed as an extension of Theorem 2.1 of [8] to the case of two ordered G -metric spaces.

Since the class of weakly compatible mappings includes R -weakly commuting mappings, Theorem 2.1 generalizes the comparable results in [8].

Corollary 2.3 *Let (X, \leq) be a partially ordered set and G_1, G_2 be two G -metrics on X such that $G_2(x, y, z) \leq G_1(x, y, z)$ for all $x, y, z \in X$ with a complete metric G_1 on X . Suppose that f, g, S and T are self-mappings on X satisfying the following properties:*

$$\begin{aligned} G_1(fx, fx, gy) &\leq a_1 G_2(Sx, Sx, Ty) + a_2 G_2(Sx, Sx, fx) + a_3 G_2(Ty, Ty, gy) \\ &\quad + a_4 [G_2(Sx, Sx, gy) + G_2(Ty, Ty, fx)] \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} G_1(fx, gy, gy) &\leq a_1 G_2(Sx, Ty, Ty) + a_2 G_2(Sx, fx, fx) + a_3 G_2(Ty, gy, gy) \\ &\quad + a_4 [G_2(Sx, gy, gy) + G_2(Ty, fx, fx)] \end{aligned} \quad (2.14)$$

for all comparable $x, y \in X$, where $a_1 + a_2 + a_3 + 2a_4 < 1$. Suppose that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and f, g are dominated mappings and S, T are dominating mappings. If, for any nonincreasing sequence $\{x_n\}$ with $y_n \leq x_n$ for all $n \in \mathbb{N}$, $y_n \rightarrow u$ implies that $u \leq x_n$ and either

(a) f, S are compatible, f or S is continuous and g, T are weakly compatible
or

(b) g, T are compatible, g or T is continuous and f, S are weakly compatible,
then f, g, S and T have a common fixed point in X . Moreover, the set of common fixed points of f, g, S and T is well-ordered if and only if f, g, S and T have one and only one common fixed point in X .

Example 2.4 Let $X = [0, 1]$ be endowed with the usual ordering and G_1, G_2 be two G -metrics on X given in [13]:

$$\begin{aligned} G_1(a, b, c) &= |a - b| + |b - c| + |c - a|, \\ G_2(a, b, c) &= \frac{1}{2} [|a - b| + |b - c| + |c - a|]. \end{aligned}$$

Define the mappings $f, g, S, T : X \rightarrow X$ as

$$fx = \frac{x}{12}, \quad gx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, \frac{1}{2}), \\ \frac{x}{6} & \text{if } x \in [\frac{1}{2}, 1), \end{cases} \quad S(x) = \frac{3x}{2}, \quad T(x) = \frac{5x}{2}$$

for all $x \in X$. Clearly, f, g are dominated mappings and S, T are dominating mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Also, f, S are compatible, f is continuous and g, T are weakly compatible. Now, for all comparable $x, y \in X$, we check the following cases:

(1) If $x, y \in [0, \frac{1}{2})$, then we have

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 3y| \leq \frac{1}{12}(x + 3y) \\ &\leq \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{9}{4}y\right) \\ &= a_2G_2(fx, fx, Sx) + a_3G_2(gy, gy, Ty) \\ &\leq a_1G_2(Sx, Sx, Ty) + a_2G_2(fx, fx, Sx) + a_3G_2(gy, gy, Ty) \\ &\quad + a_4[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]. \end{aligned}$$

(2) If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then we have

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 2y| \leq \frac{1}{12}(x + 2y) \\ &\leq \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{14}{6}y\right) \\ &= a_2G_2(fx, fx, Sx) + a_3G_2(gy, gy, Ty) \\ &\leq a_1G_2(Sx, Sx, Ty) + a_2G_2(fx, fx, Sx) + a_3G_2(gy, gy, Ty) \\ &\quad + a_4[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]. \end{aligned}$$

(3) If $y \in [0, \frac{1}{2})$ and $x \in [\frac{1}{2}, 1]$, then we have

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 3y| \leq \frac{1}{12}(x + 3y) \\ &\leq \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{9}{4}y\right) \\ &= a_2G_2(fx, fx, Sx) + a_3G_2(gy, gy, Ty) \\ &\leq a_1G_2(Sx, Sx, Ty) + a_2G_2(fx, fx, Sx) + a_3G_2(gy, gy, Ty) \\ &\quad + a_4[G_2(fx, fx, Ty) + G_2(gy, gy, Sx)]. \end{aligned}$$

(4) If $x, y \in [\frac{1}{2}, 1]$, then we obtain

$$\begin{aligned} G_1(fx, fx, gy) &= \frac{1}{12}|x - 2y| \leq \frac{1}{12}(x + 2y) \\ &\leq \frac{3}{10}\left(\frac{17}{12}x\right) + \frac{3}{10}\left(\frac{14}{6}y\right) \end{aligned}$$

$$\begin{aligned}
 &= a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\
 &\leq a_1 G_2(Sx, Sx, Ty) + a_2 G_2(fx, fx, Sx) + a_3 G_2(gy, gy, Ty) \\
 &\quad + a_4 [G_2(fx, fx, Ty) + G_2(gy, gy, Sx)].
 \end{aligned}$$

Thus (2.13) is satisfied with $a_1 = a_4 = \frac{1}{10}$ and $a_2 = a_3 = \frac{3}{10}$, where $a_1 + a_2 + a_3 + 2a_4 < 1$. Similarly, (2.14) is satisfied. Thus all the conditions of Corollary 2.3 are satisfied. Moreover, 0 is the unique common fixed point of f and g .

3 Application

Let $X = L^2(\Omega)$, the set of comparable functions on Ω whose square is integrable on Ω where $\Omega = [0, 1]$, be a bounded set in \mathbb{R} . We endow X with the partial ordered \leq given by: $x, y \in X$, $x \leq y \Leftrightarrow x(t) \leq y(t)$, for all $t \in \Omega$. We consider the integral equations

$$\begin{aligned}
 x(t) &= \int_{\Omega} q_1(t, s, x(s)) ds - c(t), \\
 y(t) &= \int_{\Omega} q_2(t, s, y(s)) ds - c(t),
 \end{aligned} \tag{3.1}$$

where $q_1, q_2 : \Omega \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $c : \Omega \rightarrow \mathbb{R}^+$, to be given continuous mappings. Recently, Abbas *et al.* [35] obtained a common solution of integral equations (3.1) as an application of their results in the setup of ordered generalized metric spaces. Here we study a sufficient condition for the existence of a common solution of integral equations in the framework of two generalized metric spaces. Define $G_1, G_2 : X \times X \times X \rightarrow \mathbb{R}^+$ by

$$\begin{aligned}
 G_1(x, y, z) &= \sup_{t \in \Omega} |x(t) - y(t)| + \sup_{t \in \Omega} |y(t) - z(t)| + \sup_{t \in \Omega} |z(t) - x(t)|, \\
 G_2(x, y, z) &= \frac{1}{2} \left[\sup_{t \in \Omega} |x(t) - y(t)| + \sup_{t \in \Omega} |y(t) - z(t)| + \sup_{t \in \Omega} |z(t) - x(t)| \right].
 \end{aligned}$$

Obviously, $G_2(x, y, z) \leq G_1(x, y, z)$ for all $x, y, z \in X$. Suppose that the following hypotheses hold:

- (i) For each $s, t \in \Omega$,

$$\int_{\Omega} q_1(t, s, u(s)) ds \leq u(s)$$

and

$$\int_{\Omega} q_2(t, s, u(s)) ds \leq u(s)$$

hold.

- (ii) There exists $r : \Omega \rightarrow \Omega$ such that

$$\int_{\Omega} |q_1(t, s, u(t)) - q_2(t, s, v(t))| dt \leq r(t) |u(t) - v(t)|$$

for each $s, t \in \Omega$ with $\sup_{t \in \Omega} r(t) \leq k$ where $k \in [0, 1)$.

Then the integral equations (3.1) have a common solution in $L^2(\Omega)$.

Proof Define $fx(t) = \int_{\Omega} q_1(t, s, x(t)) dt - c(t)$ and $gx(t) = \int_{\Omega} q_2(t, s, x(t)) dt - c(t)$. As $fx(t) \leq x(t)$ and $gx(t) \leq x(t)$, so f and g are dominated maps. Now, for all comparable $x, y \in X$,

$$\begin{aligned} G_1(fx, fx, gy) &= 2 \sup_{t \in \Omega} |fx(t) - gy(t)| \\ &= 2 \sup_{t \in \Omega} \left| \int_{\Omega} q_1(t, s, x(t)) dt - \int_{\Omega} q_2(t, s, y(t)) dt \right| \\ &\leq 2 \sup_{t \in \Omega} \int_{\Omega} |q_1(t, s, x(t)) - q_2(t, s, y(t))| dt \\ &\leq 2 \sup_{t \in \Omega} r(t) |x(t) - y(t)| \\ &\leq 2k \sup_{t \in \Omega} |x(t) - y(t)| \\ &= kG_2(x, y, y) \\ &\leq k \max \{ G_2(x, x, y), G_2(fx, fx, x), G_2(gy, gy, y), \\ &\quad [G_2(fx, fx, y) + G_2(gy, gy, x)]/2 \}. \end{aligned}$$

Similarly,

$$\begin{aligned} G_1(fx, gy, gy) &\leq k \max \{ G_2(x, y, y), G_2(fx, x, x), G_2(gy, y, y), \\ &\quad [G_2(fx, y, y) + G_2(gy, x, x)]/2 \} \end{aligned}$$

is satisfied. Now we can apply Theorem 2.1 by taking S and T as identity maps to obtain the common solutions of integral equations (3.1) in $L^2(\Omega)$. \square

Remarks

(1) If we take $f = g$ in Theorem 2.1, then it generalizes Corollary 2.3 in [8] to a more general class of commuting mappings in the setup of two ordered G -metric spaces.

(2) If we take $S = T$ in Theorem 2.1, then Corollary 2.4 in [8] is a special case of Theorem 2.1.

(3) If $S = T = I_X$ (: the identity mapping on X) in Theorem 2.1, then we obtain Corollary 2.5 in [8] in a more general setup.

(4) Corollary 2.6 of [8] becomes a special case of Theorem 2.1 if we take $f = g$ and $S = T = I_X$.

(5) A G -metric naturally induces a metric d_G given by $d_G(x, y) = G(x, y, y) + G(x, x, y)$. If the G -metric is not symmetric, then the inequalities (2.1), (2.2), (2.13) and (2.14) do not reduce to any metric inequality with the metric d_G . Hence our results do not reduce to fixed point problems in the corresponding metric space (X, \leq, d_G) .

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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